Published by Institute of Physics Publishing for SISSA
Received: *August 18, 2005*
Accepted: *August 24, 2005*
Published: *September 20, 2005*

D-branes and discrete torsion II

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Abstract: We derive D-brane gauge theories for $\mathbb{C}^3/\mathbb{Z}_n \times \mathbb{Z}_n$ orbifolds with discrete torsion and study the moduli space of a D-brane at a point. We show that, as suggested in previous work, closed string moduli do not fully resolve the singularity, but the resulting space — containing $n-1$ conifold singularities — is somewhat surprising. Fractional branes also have unusual properties.
We also define an index which is the CFT analog of the intersection form in geometric compactification, and use this to show that the elementary D6-brane wrapped about $T^6/\mathbb{Z}_n \times \mathbb{Z}_n$ must have U(n) world-volume gauge symmetry.

Keywords: D-branes, Superstring Vacua.

Contents

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1. Introduction

String theory can be well defined on spaces that from the classical geometric point of view have singularities. In recent years, D-branes have brought a new perspective on this.

In this work we will study D-branes at a $\mathbb{C}^3/\mathbb{Z}_n \times \mathbb{Z}_n$ singularity with discrete torsion, following the ideas of [1−3]. Discrete torsion was defined in [1] for orbifolds M/Γ in world-sheet terms: different sectors in the closed string path integral distinguished by their twisted boundary conditions are weighed by phases. The first example is genus one with two twists g and h corresponding to the two generators of π_1 and a weight $\epsilon(g,h)^2$. Higher loop modular invariance requires $\epsilon(g,h)$ to be a cocycle in $H^2(\Gamma, \mathrm{U}(1))$. For $\Gamma = \mathbb{Z}_n \times \mathbb{Z}_n$ this is $\mathbb{Z}_n$ and we study the case where $\epsilon(g,h)$ is a generator of this group ("minimal discrete torsion").

Singularities can appear as degenerations in Calabi-Yau manifolds and from the point of view of algebraic geometry admit two types of desingularization. Singularities caused by degeneration of the complex structure can be deformed, while singularities caused by degeneration of the Kähler form can be resolved (or blown up). As part of a compact space the possible resolutions depend on the global topology of the space: harmonic $(2,1)$-forms correspond to complex structure deformations, and harmonic $(1,1)$-forms to Kähler form deformations. The analysis of the conformal field theory of strings on $\mathbb{C}^3/\mathbb{Z}_n \times \mathbb{Z}_n$ uses standard orbifold techniques and we review these results. While string theory without discrete torsion allows only Kähler deformations, the theory with minimal discrete torsion allows only complex structure deformations. In the latter case, although conformal field theory does not lead to a clear geometric picture of these deformations, one can make hypotheses based on parameter counting and other general considerations which strongly suggest that the complex structure deformations provided by the string theory do *not* allow completely resolving the singularity. In [2] it was suggested that the $n = 2$ model contains an irresolvable conifold singularity, while the $n = 3$ model could contain an irresolvable singularity of codimension 2.

By studying a D-brane in this background, one sees space-time arise in a different way. The low energy motion of D-branes is described by a gauge theory, and space-time appears as its moduli space. The two types of deformation are then obtained as modifications of the F-flatness conditions (complex structure) and of the D-flatness conditions (Kähler form).

World-volume theories for D-branes at orbifold singularities can be obtained as projections of $\mathcal{N} = 4$ super Yang-Mills theory and in [3] it was argued that discrete torsion is implemented by doing this with projective representations of Γ, and the $\mathbb{C}^3/\mathbb{Z}_2 \times \mathbb{Z}_2$ case was studied. The resulting moduli space was exactly as predicted in [2]: after turning on all moduli, an isolated singularity remains.

In the present paper we extend this to minimal discrete torsion for $\mathbb{C}^3/\mathbb{Z}_n \times \mathbb{Z}_n$ and arbitrary n. We find that for arbitrary n, after turning on all the available complex structure deformations, the three fixed lines are resolved, but $n - 1$ conifold singularities appear.

In section 2 we present the orbifolds and find the closed string marginal operators for various choices of discrete torsion. In section 3 we derive the gauge theory, in section 4 describe the generic branch of moduli space, and in section 5 study special branches which appear at partial resulutions.

The interpretation of fractional branes is discussed in section 6. There are many differences from the case without discrete torsion; some puzzles are listed, including one on charge quantization. We point out that charge quantization is best formulated in terms of an index, $\mathrm{Tr}_{ab}\ (-1)^F$ in the open string sector with boundary conditions a and b, and resolve the puzzle in this case. Section 7 summarizes the conclusions.

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2. $\mathbb{C}^3/\mathbb{Z}_n \times \mathbb{Z}_n$ orbifolds and discrete torsion

In this section we introduce the orbifolds we are going to study and discuss their possible resolutions. Let $\mathbb{C}^3$ be described by three complex coordinates, z_1, z_2, z_3. The group $\Gamma = \mathbb{Z}_n \times \mathbb{Z}_n$ is generated by two elements g_1 and g_2. We will denote a generic element $g_1^a g_2^b$ of Γ as (a, b).

As an action of $\mathbb{Z}_n \times \mathbb{Z}_n$ on $\mathbb{C}^3$, we take $R(g)$ defined by

$$\begin{aligned} g_1 &: (z_1, z_2, z_3) \rightarrow (z_1, e^{-\frac{2\pi i}{n}} z_2, e^{\frac{2\pi i}{n}} z_3) \\ g_2 &: (z_1, z_2, z_3) \rightarrow (e^{\frac{2\pi i}{n}} z_1, z_2, e^{-\frac{2\pi i}{n}} z_3). \end{aligned} \tag{2.1}$$

This is the unique faithful representation in the sense that any $R'(g) = R(H(g))$ for some group homomorphism H.

The 2-cocycle classes of $H^2(\Gamma, \mathrm{U}(1)) \cong \mathbb{Z}_n$ are represented by

$$\begin{aligned} \epsilon^m(g, h) : \Gamma \;\times\; \Gamma &\rightarrow \mathrm{U}(1) \\ \big((a, b), (a', b')\big) &\rightarrow \zeta^{m(ab' - a'b)}\,, \end{aligned} \tag{2.2}$$

where $\zeta = e^{\frac{\pi i}{n}}$ for n even and $\zeta = e^{\frac{2\pi i}{n}}$ for n odd. The different values of $m = 0, \ldots, n-1$ correspond to the different elements of $H^2(\Gamma, \mathrm{U}(1)) \cong \mathbb{Z}_n$. In this paper we will study the theories that arise when $(m, n) = 1$, that is, when ζ^{2m} is a generator of $\mathbb{Z}_n$.

2.1 Possible resolutions of $\mathbb{C}^3/\mathbb{Z}_n \times \mathbb{Z}_n$

One can define $\mathbb{C}^3/\mathbb{Z}_n \times \mathbb{Z}_n$ as an affine variety in $\mathbb{C}^4$, given by $F(x, y, z, t) = 0$, where $F(x, y, z, t) = xyz - t^n$. The Γ-invariant variables are $m_i = (z_i)^n$ and $b = z_1 z_2 z_3$, related by $m_1 m_2 m_3 = b^n$. This variety presents non-isolated singularities, in the form of three (complex) fixed lines of $\mathbb{C}^2/\mathbb{Z}_n$ singularities: $(z_1, 0, 0)$, $(0, z_2, 0)$ and $(0, 0, z_3)$. These three lines intersect at the origin, that is a fixed point under the action of the whole group Γ.

Using toric geometry it is fairly straightforward to give a description of the possible blow-up resolutions of $\mathbb{C}^3/\mathbb{Z}_n \times \mathbb{Z}_n$. The fact that the singularities are non-isolated does not affect the discussion at this level. The discussion parallels the one for $\mathbb{C}^2/\mathbb{Z}_n$ [4], so we will be brief. The fan for $\mathbb{C}^3/\mathbb{Z}_n \times \mathbb{Z}_n$ has a single big cone, determined by three vectors $(0, 0, 1)$, $(n, 0, 1)$ and $(0, n, 1)$. The first intersection (away from the origin) of the three edges of the cone with the lattice determines the hyperplane $z = 1$, so the singularities are Gorenstein. The intersection is a triangle in the hyperplane, with vertices $(0, 0, 1)$, $(n, 0, 1)$ and $(0, n, 1)$. The resolution is far from unique: the possible blow-up resolutions are represented by different triangulations of this triangle. For a given n there are many possible triangulations, all related by flop transitions.

In [5], the case $n = 2$ without discrete torsion was studied using D-brane methods. As we discuss below the moduli are Kähler moduli and it was shown using toric methods that all the possible resolutions can be obtained from the D-brane gauge theory.

Another way to resolve the orbifold is to deform its complex structure. To find the possible relevant deformations, first we compute the ideal generated by the partial derivatives of F: $J = \{\partial_i F\} = \{yz, xz, xy, t^{n-1}\}$. The quotient of the ring of polynomials in

x, y, z, t by J gives us the set of relevant deformations of F:

$$Q = \frac{C[x,y,z,t]}{\{\partial_i F\}} = \left\{x^a, y^b, z^c, 1, t, \ldots, t^{n-2}\right\}. \tag{2.3}$$

The quotient ring is infinite dimensional, something that matches with the fact that we have non-isolated singularities. If we deform F adding a power of x, we resolve a complex line (and similarly for y and z). If we deform by 1 or t, we resolve completely the variety. If we resolve by t^r with $1 < r \leq n-2$ we reduce the singularities to those of $xyz - t^r$.

We see that mathematically these orbifolds can be completely resolved by deformation of their complex structure. This does not necessarily mean that all of these deformations are available in the physical theory — at the very least, each physical deformation (modulus) must correspond to a marginal operator in closed string theory on the orbifold. Thus we need to compute this spectrum.

2.2 Cohomology

Space-time supersymmetry relates marginal operators to ground states in the Ramond sector. As is well known, for sigma model compactification these are in one-to-one correspondence with the cohomology of the target space, so if the orbifold is the limit of a smooth target space M this computation will give us $H^*(M)$. We will also think of the results as defining the cohomology of the orbifold. As we are about to see, it depends markedly on whether we introduce discrete torsion or not.

We follow the procedure used in [2] (see also [6]). There are n^2 sectors in the theory: one untwisted sector corresponding to the element $(0,0)$ and n^2-1 twisted sectors corresponding to the remaining elements of Γ. For each sector g, we let M_g be the subset of $\mathbb{C}^3$ fixed under the action of g. In the ordinary case, without discrete torsion, the usual sigma model analysis generalizes to show that the ground states in the g-twisted sector correspond to forms $H^{p,q}(M_g)$ that are invariant under the action of Γ:

$$R(h)w = w \qquad \forall h \in \Gamma\,. \tag{2.4}$$

Such a form $\omega^{p,q}$ will contribute to $H^{p+s,q+s}(M)$ with s a computable function of g: for g acting as $z_i \to e^{i\theta_i} z_i$ with $0 \leq \theta_i < 2\pi$, it is given by

$$s = \sum_i \frac{\theta_i}{2\pi}\,. \tag{2.5}$$

This shift is derived for example in [6] as the shift in the fermion number and U(1) charges of the vacuum for the twisted sector g.

Let us start with the untwisted sector $(0,0)$. The fixed point set in this case is of course $\mathbb{C}^3$ itself, but not all the forms defined on $\mathbb{C}^3$ are Γ-invariant. For instance $dz_1 \wedge d\bar{z}_1$ is invariant and it is kept, but $dz_1 \wedge dz_2$ is not invariant and it is projected out. The contribution to the Hodge diamond coming from the untwisted sector is, for $n > 2$

$$\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & 3 & & 0 & \\
1 & & 0 & & 0 & & 1 \\
 & 0 & & 3 & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 1 & & &
\end{array}$$

For $n = 2$ the untwisted sector has instead $h^{2,1} = h^{1,2} = 3$, since for instance the form $dz_1 \wedge dz_2 \wedge d\bar{z}_3$ survives.

Now let's consider the $n^2 - 1$ twisted sectors. We are going to discuss the twisted sector corresponding to the element (a, b). First we have to look for the fixed point set of the element (a, b). There are two classes of elements, according to their fixed point sets. The $3(n-1)$ elements of the form $(a, 0)$, $(0, b)$ and (a, a) have a (complex) line of fixed points: $(z_1, 0, 0)$, $(0, z_2, 0)$ and $(0, 0, z_3)$ respectively, as can be seen from (2.1). The remaining $(n-1)(n-2)$ elements have only the origin as their fixed point set.

As a warm-up, we compute the contribution of the twisted sectors for the theories without discrete torsion. For each of the $3(n-1)$ elements of the kind $(a, 0)$, $(0, b)$ and (a, a), the cohomology before projecting is generated by $1, dz, d\bar{z}$ and $dz \wedge d\bar{z}$. Without discrete torsion 1 and $dz \wedge d\bar{z}$ are Γ-invariant, whereas dz and $d\bar{z}$ are not. The shift for these $3(n-1)$ elements is easily computed to be $s = 1$ for all of them, so in the case of theories without discrete torsion, the contribution of these $3(n-1)$ twisted sectors to the Hodge diamond is $h^{1,1} = h^{2,2} = 3(n-1)$. For the remaining $(n-1)(n-2)$ elements, the origin is the only fixed point, and in absence of discrete torsion, the $(0,0)$-form 1 is Γ-invariant. After the shift, half of these elements contribute to $H^{1,1}$, and the other half contribute to $H^{2,2}$. All in all, in absence of discrete torsion, the contribution of the twisted sectors to the Hodge diamond is

$$\begin{array}{ccccccc}
 & & & 0 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & \frac{(n+4)(n-1)}{2} & & 0 & \\
0 & & 0 & & 0 & & 0 \\
 & 0 & & \frac{(n+4)(n-1)}{2} & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 0 & & &
\end{array}$$

After this warm-up, let's turn to the theories with discrete torsion. The whole discussion depends on $r = \gcd(m, n)$, and the results differ markedly between the cases $r = 1$ (m and n are relative primes) and $r > 1$. Now in the sector g we keep the forms w such that

$$(\epsilon(g, h))^2 \; R(h)w = w \qquad \forall h \in \Gamma\,. \tag{2.6}$$

Since $\epsilon(1, h) = 1 \; \forall h$, the untwisted sector remains the same, so we go to the twisted sectors. Let's deal first with the $3(n-1)$ sectors corresponding to elements that fix a complex line. Recall that before projecting the cohomology is given by $1, dz, d\bar{z}$ and $dz \wedge d\bar{z}$.

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For 1 and $dz \wedge d\bar{z}$, $R(h)w = w$, so they survive if $(\epsilon(g,h))^2 = 1$, and this happens for $3(r-1)$ elements of these sectors. Furthermore, if $r = 1$ there are exactly 3 elements for which dz is kept and 3 elements for which $d\bar{z}$ is kept. If $r > 1$ then dz and $d\bar{z}$ are projected out. As before, the shift for these elements is 1, but now the forms that we keep are in $H^{0,1}$ and $H^{1,0}$ of the respective fixed point sets, so they contribute to $H^{1,2}$ and $H^{2,1}$ of the whole orbifold. Finally, we have to consider the remaining twisted sectors, associated to elements that only leave fixed the origin. Now the only element in the cohomology is 1, and it is kept by $(r-1)(r-2)$ elements, half of them contributing after the shift to $h^{1,1}$ and half of them to $h^{2,2}$.

Putting together these facts, the twisted sector contribution to the Hodge diamond for the $r = 1$ case is:

$$\begin{array}{ccccccc}
 & & & 0 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & 0 & & 0 & \\
0 & & 3 & & 3 & & 0 \\
 & 0 & & 0 & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 0 & & &
\end{array}$$

and for $r > 1$

$$\begin{array}{ccccccc}
 & & & 0 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & \frac{(r+4)(r-1)}{2} & & 0 & \\
0 & & 0 & & 0 & & 0 \\
 & 0 & & \frac{(r+4)(r-1)}{2} & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 0 & & &
\end{array}$$

When $r > 1$ the effect of discrete torsion is just to decrease the number of ground states in $h^{1,1}$ and $h^{2,2}$ compared to the case without discrete torsion (that can be considered a particular case, $m = 0$, $r = n$). On the other hand, when $r = 1$ there are no Kähler deformations available, and we find the same number of complex deformations for every n. Thus the general $\mathbb{Z}_n \times \mathbb{Z}_n$ singularity will have either complex structure or Kähler deformations, but not both.

3. The D-brane gauge theories

In this section we derive the gauge theory that describes D-branes on the $\mathbb{C}^3/\mathbb{Z}_n \times \mathbb{Z}_n$ orbifold with discrete torsion for $\gcd(m,n) = 1$. After presenting the gauge theory, we look for its moduli space and check that, for the regular representation, it reproduces the original orbifold. We then include the modification of the superpotential due to the 3 possible complex structure deformations that we found in the previous section, and determine the corresponding deformed moduli space. As we will see, the addition of the complex structure deformations resolves the fixed lines of the original orbifold, leaving $n-1$ conifold singularities.

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3.1 Derivation of the quotient theory

The general procedure is by now well known [7]. There are two choices that we have to make: the action of Γ on space-time, $R(g)$, and the action of Γ on the Chan-Paton indices, $\gamma(g)$. If we want to describe N independent D-branes on the orbifold, we start by placing $|\Gamma|N$ D-branes on the original $\mathbb{C}^3$. Initially this corresponds to a U($|\Gamma|N$) SYM theory with $\mathcal{N}=4$ in $d=4$. To construct the quotient gauge theory we impose projection conditions for the gauge and the scalar fields

$$\begin{aligned} A &= \gamma(g)A\gamma(g)^{-1} \\ Z^i &= R^{ij}(g)\left(\gamma(g)Z^j\gamma(g)^{-1}\right). \end{aligned} \tag{3.1}$$

Plugging back the fields surviving these projections into the original U($|\Gamma|N$) theory, we obtain the quotient gauge theory.

Our choice of the space-time action is (2.1); since this lies in SU(3), the corresponding theory will have $\mathcal{N}=1$ in $d=4$. Where does the discrete torsion enter the game? As observed in [3], it is implemented in the gauge theory by taking the action on the Chan-Paton indices to be a *projective* representation of Γ, that is $\gamma(g)\gamma(h)=\epsilon(g,h)\gamma(gh)$.

According to [8], when $\epsilon \equiv \zeta^{2m}$ is a primitive n'th root of 1 (true when $\gcd(m,n)=1$), $\mathbb{Z}_n \times \mathbb{Z}_n$ has a unique irreducible projective representation. It is

$$\gamma_1(g_1) = P \qquad \gamma_1(g_2) = Q\,, \tag{3.2}$$

where the matrices P and Q are as follows. For n odd,

$$P = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 & 1 \\ 1 & 0 & 0 & \dots & 0 \end{pmatrix} \qquad Q = \begin{pmatrix} 0 & \epsilon & 0 & \dots & 0 \\ 0 & 0 & \epsilon^2 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 & \epsilon^{n-1} \\ 1 & 0 & 0 & \dots & 0 \end{pmatrix}. \tag{3.3}$$

For n even, P is as above, define $\delta^2=\epsilon$ and

$$Q = \begin{pmatrix} 0 & \delta & 0 & \dots & 0 \\ 0 & 0 & \delta^3 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 & \delta^{2n-3} \\ \delta^{2n-1} & 0 & 0 & \dots & 0 \end{pmatrix}. \tag{3.4}$$

Note that $PQ=\epsilon QP$, so although Γ is abelian, the projective representation is not. The general representation is a direct sum of M copies of this:

$$\gamma_M = \gamma_1 \otimes \mathbf{1}_M\,.$$

In particular, the regular representation of dimension n^2 is γ_n.

We can now give the solution of the projection (3.1) for every m such that $(m,n)=1$. By composing the space-time action (2.1) with a $\mathbb{Z}_n \times \mathbb{Z}_n$, the general solution based on the irreducible representation (3.2) can be brought to the form

$$A = \mathrm{I} \qquad Z_1 = P \qquad Z_2 = Q \qquad Z_3 = (PQ)^{-1}\,. \tag{3.5}$$

The most general solution is obtained by tensoring this $n\times n$ solution with $M\times M$ matrices.

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Intuitively we expect a theory describing N 'true' branes away from the origin will involve $|\Gamma|N = n^2N$ images and thus use N copies of the regular representation, i.e. γ_M with $M = Nn$.[1] On the other hand string consistency conditions do not require M to be a multiple of n; we will return to the interpretation of this possibility later.

After tensoring (3.5) with $M \times M$ matrices and substituting into the the U(Mn) SYM theory, we obtain the quotient theory, which we proceed to describe.

3.2 The orbifold theory

The gauge theory describing $\mathbb{C}^3/\mathbb{Z}_n \times \mathbb{Z}_n$ with minimal discrete torsion is a U(M) theory with three chiral superfields ϕ_i, $i = 1, 2, 3$ in the adjoint. The superpotential is

$$W = \mathrm{tr}\{\phi_1(\phi_2\phi_3 - \epsilon^{-1}\phi_3\phi_2)\}\,. \tag{3.6}$$

This leads to the F-flatness conditions

$$\phi_i\phi_j - \epsilon^{-1}\phi_j\phi_i = 0 \qquad i \neq j \tag{3.7}$$

and D-flatness conditions

$$\sum_i [\phi_i, \phi_i^\dagger] = 0\,. \tag{3.8}$$

The superpotential (3.6) preserves a U(1)^3 subgroup of the SU(4) R-symmetry of $\mathcal{N} = 4$ super Yang-Mills, the individual phase rotations of the ϕ_i. The diagonal U(1) is the usual $\mathcal{N} = 1$ R-symmetry.

So far, we haven't introduced in the gauge theory the three complex structure deformations that we found in the previous section. These appear as the following deformation of the superpotential of the gauge theory:

$$\Delta W = \sum_i \zeta_i \,\mathrm{tr}\, \phi_i\,. \tag{3.9}$$

This is consistent with the U(1)^3 symmetry if we assign ζ_1 the charges $(0, 1, 1)$, and similarly for ζ_2 and ζ_3. The symmetry will then prohibit corrections higher order in ϕ.[2]

The new term modifies the F-flatness conditions to

$$\begin{aligned}
\phi_1\phi_2 - \epsilon^{-1}\phi_2\phi_1 &= -\zeta_3\\
\phi_2\phi_3 - \epsilon^{-1}\phi_3\phi_2 &= -\zeta_1\\
\phi_3\phi_1 - \epsilon^{-1}\phi_1\phi_3 &= -\zeta_2\,.
\end{aligned} \tag{3.10}$$

The D-flatness conditions (3.8) remain the same. Our goal will be to find the new moduli space, corresponding to these deformed conditions.

[1] There is a simple argument why one expects to need $M \geq n$ to get a three complex dimensional moduli space. One can see that supersymmetric vacua (in the undeformed orbifold theory) are still described by *commuting* matrices in the original U(Mn) theory; to make commuting matrices out of the solution (3.5) one must tensor them with another projective representation with the opposite cocycle, which will have dimension $M \geq n$.

[2] It can be checked by world sheet computation that the disk diagram with a bulk insertion of the twisted closed string operator $V(\zeta_i)$ and a boundary insertion of $V(F_i)$, the auxiliary field in the χ_i multiplet, is non-zero. The U(1)^3 symmetry is the unbroken subgroup of SO(6) rotations around the fixed point and one can also check that $V(\zeta_i)$ has the charges stated above.

4. Moduli spaces as varieties

In our discussion of the moduli space, we will use the gauge invariant polynomials[3]

$$M_{ij\ldots k} = \frac{1}{n}\,\mathrm{tr}\,\{\phi_i\phi_j\ldots\phi_k\}\,. \tag{4.1}$$

Taking the trace of the F-flatness conditions (3.10) we find

$$M_{ij} = -\left(1-\epsilon^{-1}\right)^{-1}\zeta_k \equiv \xi_k\,. \tag{4.2}$$

Let us first solve the F and D conditions in the undeformed theory. We start with the regular representation. For $M = n$ (that is, $N = 1$, a single D-brane), recalling that $PQ = \epsilon QP$, we see that a solution is given by

$$\phi_1 = z_1 Q \qquad \phi_2 = z_2 P \qquad \phi_3 = z_3(QP)^{-1} \tag{4.3}$$

What is the equation for the moduli space? If we define n_1 as the number of ϕ_1's in a given gauge invariant polynomial $M_{ij\ldots k}$, and similarly for n_2 and n_3, the only non-zero gauge invariant polynomials have $\mathrm{U}(1)^3$ charges $n_1 \equiv n_2 \equiv n_3$ (n). In particular

$$\begin{aligned}
M_{\underbrace{11\ldots1}_{n}} &= z_1^n\\
M_{\underbrace{22\ldots2}_{n}} &= z_2^n\\
M_{\underbrace{33\ldots3}_{n}} &= (-1)^{n-1}z_3^n\\
M_{123} &= z_1z_2z_3
\end{aligned} \tag{4.4}$$

so it is natural to identify these gauge invariant polynomials with the Γ-invariant variables m_i and b of $\mathbb{C}^3/\mathbb{Z}_n\times\mathbb{Z}_n$, and conclude that the undeformed moduli space is

$$M_{\underbrace{11\ldots1}_{n}}M_{\underbrace{22\ldots2}_{n}}M_{\underbrace{33\ldots3}_{n}} = (-1)^{n-1}(M_{123})^n\,. \tag{4.5}$$

From now on, we drop the brackets, and unless otherwise stated, it is understood that $M_{ii\ldots i}$ has n indices.

What happens if we don't take the regular representation? For $1 \le M < n$, taking determinants on both sides of the original F-flatness conditions (3.7) and recalling that ϵ is a primitive n*th*-root of 1, we readily see that $|\phi_i||\phi_j| = 0$, so at least two out of three matrices have zero determinant. By going to a basis where one of these matrices with zero determinant, say ϕ_1, is diagonal, we can easily argue that most of the off diagonal elements of ϕ_2 and ϕ_3 have to vanish in order to satisfy the F-flatness conditions, and then, for $M < n$, the remaining off diagonal elements have to vanish in order to satisify the D-flatness conditions. Therefore, for $M < n$, all the solutions of the F and D flatness conditions are diagonal matrices, and there are no Higgs branches. For $M = 1$, the moduli space is given by the three lines $M_1 = M_2 = 0$, $M_2 = M_3 = 0$ and $M_1 = M_3 = 0$.

[3] Our normalizations differ from those of [3].

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We turn to the deformed case. For the regular representation, there is a simple three parameter family of solutions of the deformed F-flatness conditions (3.10):

$$\phi_1 = z_1 Q + \xi_3 \frac{P^{-1}}{z_2} \qquad \phi_2 = z_2 P + \xi_1 \frac{QP}{z_3} \qquad \phi_3 = z_3 (QP)^{-1} + \xi_2 \frac{Q^{-1}}{z_1}\,. \tag{4.6}$$

These do not satisfy the D-flatness conditions and it appears to be quite hard to find explicit solutions of both conditions. On the other hand we are guaranteed by general results that, for every solution of the F-flatness conditions, there will exist a unique gauge equivalence class of solutions of the D-flatness conditions [9]. In the present case it is defined by minimizing the functional $f = \mathrm{Tr} \sum_i \phi_i \phi_i^+$ over a gauge orbit $\phi_i \to g \phi_i g^{-1}$ with $g \in GL(n)$ the complexified gauge group. We do not need to find these solutions explicitly in order to describe the moduli space with its complex structure; they would have been useful if (for example) we wanted to compute the metric.

For many purposes it is more useful to describe the moduli space as a variety, i.e. as the solutions of a set of polynomial equations between gauge invariant polynomials. In particular this will bypass the need to determine which of the solutions (4.6) are gauge equivalent. An overcomplete set of such equations is obtained by combining all equations $\mathrm{Tr}\, W'(\phi) P(\phi) = 0$ for all polynomials P with the complete set of identities on gauge-invariant polynomials constructed from matrices of dimension n. The latter are quite simple for $n = 2$ and this approach was followed in [3]; however for $n > 2$, the system of relations between gauge invariant polynomials is very complicated.

The observation that makes the problem tractable is that, if we are satisfied just to study the moduli space of the gauge theory, then relations valid *on the moduli space* are good enough for us. In particular, the F-flatness conditions (3.10) turn out to be a powerful constraint that allows us to establish many relations among the gauge invariant polynomials that, though not valid in general, hold on the moduli space. In fact, we can argue that on the moduli space of this $\mathrm{U}(n)$ gauge theory, when all $\zeta \neq 0$, any non-zero gauge invariant polynomial can be written in terms of the 4 gauge invariant polynomials used to define the undeformed moduli space! The proof of this statement is presented in the appendix.

Accepting this result, we conclude that the deformed moduli space is indeed a variety in $\mathbb{C}^4$. As we argued the branch we found is three dimensional and by general results in algebraic geometry a three dimensional subvariety must be a hypersurface; i.e. the solution of a polynomial equation in $\mathbb{C}^4$. This equation must respect the $\mathrm{U}(1)^3$ symmetry acting on ϕ_i and ξ_i as well as an obvious permutation symmetry.

It is natural to suppose that this equation would be a deformation of (4.5) with corrections polynomial in the ξ's. Adding all polynomial perturbations respecting the symmetries we arrive at the ansatz

$$M_{11\dots 1} M_{22\dots 2} M_{33\dots 3} + c(M_{11\dots 1}\xi_1^n + M_{22\dots 2}\xi_2^n + M_{33\dots 3}\xi_3^n) = \sum_{k=0}^{[n/2]} a_k M_{123}^{n-2k} (\xi_1\xi_2\xi_3)^k \tag{4.7}$$

for the equation determining this branch of moduli space.

We can determine the coefficients by requiring that the solutions (4.6) are solutions of (4.7). The corresponding invariants are

$$\begin{aligned} M_{11\ldots1} &= z_1^n + \frac{\xi_3^n}{z_2^n} \\ M_{22\ldots2} &= z_2^n + (-1)^{n-1}\frac{\xi_1^n}{z_3^n} \\ M_{33\ldots3} &= (-1)^{n-1} z_3^n + \frac{\xi_2^n}{z_1^n} \\ M_{123} &= z_1 z_2 z_3 + \epsilon^{-1}\frac{\xi_1\xi_2\xi_3}{z_1 z_2 z_3} \end{aligned} \tag{4.8}$$

and plugging these into the proposed relation (4.7), we find

$$c = -1\,; \qquad a_k = (-1)^{n+k-1}\epsilon^{-k}\frac{n}{n-k}\begin{pmatrix} n-k \\ k \end{pmatrix}. \tag{4.9}$$

Now that we have checked the equation (4.7) on the explicit solutions (4.6), we can assert that it *is* the equation defining this branch of moduli space; the hypothesis that it might take the form (4.7) has been explicitly verified. In general the equations defining the moduli space derived by following the general procedure could have additional solutions which will be additional branches of the moduli space. Let us focus for now on this branch.

The coefficients a_k correspond to the n'th Chebyshev polynomial of the first kind: $T_n(\cos\theta) = \cos n\theta$, and thus the equation describing this branch of the moduli space takes a simple form: it is $F(M_{ii\ldots i}, M_{123}) = 0$ where

$$\begin{aligned} F(M_{11\ldots1}, M_{22\ldots2}, M_{33\ldots3}, M_{123}) &= M_{11\ldots1}M_{22\ldots2}M_{33\ldots3} - M_{11\ldots1}\xi_1^n - M_{22\ldots2}\xi_2^n - \\ &\quad - M_{33\ldots3}\xi_3^n - 2(-1)^{n-1}\chi^n T_n\left(\frac{M_{123}}{2\chi}\right), \end{aligned} \tag{4.10}$$

where we defined $\chi = (\epsilon^{-1}\xi_1\xi_2\xi_3)^{1/2}$.

This is not the general deformation of (4.5) and thus this variety might generically be singular. Singularities will be solutions of $\partial F/\partial M_{ii\ldots i} = \partial F/\partial M_{123} = F = 0$. To start with, $\partial F/\partial M_{ii\ldots i} = 0$ requires that

$$M_{ii\ldots i} = +\left(\frac{\xi_j\xi_k}{\xi_i}\right)^{n/2} \tag{4.11}$$

or

$$M_{ii\ldots i} = -\left(\frac{\xi_j\xi_k}{\xi_i}\right)^{n/2} \tag{4.12}$$

so at the singularity $F(M_{ii\ldots i}, M_{123}) = 2(\xi_1\xi_2\xi_3)^{\frac{n}{2}}\left(\mp1 - T_n(M_{123}/2\chi)\right)$. We have to determine if any of the $n-1$ solutions of $\frac{\partial F}{\partial M_{123}} = 0$ is also a solution of $F = 0$ for these values of $M_{ii\ldots i}$. Defining $\cos\theta = (M_{123}/2\chi)$, the $n-1$ roots of $\frac{\partial F}{\partial M_{123}}$ are given by

$$\theta_k = \frac{\pi k}{n} \qquad k = 1,\ldots,n-1 \tag{4.13}$$

and since $T_n(\cos\theta_k) = (-1)^k$, we see that the variety has $n-1$ singularities:

$$M_{ii\ldots i} = (-1)^{k-1}\left(\frac{\xi_j\xi_k}{\xi_i}\right)^{n/2} \qquad M_{123} = 2\chi cos\frac{\pi k}{n} \qquad k = 1,\ldots,n-1\,. \tag{4.14}$$

It is easy to check that these are conifold singularities. To do so, expand (4.10) around any of the singularities $M_{ii\ldots i} = M^r_{ii\ldots i} + x_i$ and $M_{123} = M^r_{123} + t$, where $M^r_{ii\ldots i}$ and M^r_{123} are the values at the r'th singularity (4.14) . The result is

$$\left(\frac{\chi^{n-2}(-1)^{r-1}n^2}{4\sin^2\theta_r}\right)t^2 + \mathcal{O}(t^3) = \sum_{i<j} M^r_{kk\ldots k}\, x_i x_j + \mathcal{O}(x^3) \tag{4.15}$$

and since the determinant of this quadratic form is different from zero, locally the remaining singularities are conifold singularities.

5. More on moduli spaces

The analysis of the previous section treated the generic branch of moduli space, but there can also be special branches. The discussion depends very much on how many $\zeta \neq 0$ so we discuss each case separately. In addition we consider the moduli spaces for $M < n$ which are also physically relevant. Finally, we will also be interested in the Coulomb branch of moduli space. This is defined in the p-brane theories with $p < 3$ obtained by naive dimensional reduction as follows: this theory contains scalar partners of the gauge field, which we can write as real adjoint fields X^i. Their potential will be minimized by any vevs with $[X^i, X^j] = 0$ and if $\zeta = 0$ and $\phi = 0$ this is generically a supersymmetric vacuum with unbroken $\mathrm{U}(1)^n$. The Coulomb branch of interest to us is then the moduli space of such vacua with general ϕ_i.

5.1 The undeformed moduli space

In the orbifold limit, when we set ξ_i to zero, the equation that describes the generic branch is

$$M_{11\ldots 1}M_{22\ldots 2}M_{33\ldots 3} = (-1)^{n-1}M^n_{123}\,. \tag{5.1}$$

This moduli space has three fixed (complex) lines of singularities, all three with $M_{123} = 0$: $M_{11\ldots 1} = M_{22\ldots 2} = 0$, $M_{22\ldots 2} = M_{33\ldots 3} = 0$ and $M_{11\ldots 1} = M_{33\ldots 3} = 0$. These three fixed lines correspond to the three lines of $\mathbb{C}^2/\mathbb{Z}_n$ singularities of $\mathbb{C}^3/\mathbb{Z}_n \times \mathbb{Z}_n$.

The fixed lines also are in correspondence with the moduli space for $M = 1$. For $M = 1$ the F-flatness conditions become $\phi_1\phi_2 = \xi_3$ (and permutations) and for $\xi_i = 0$ the solutions are just the fixed lines. This also generalizes to $M > 1$: the Coulomb branch allows a single non-zero ϕ_i in each of the U(1) factors and i can be different in each factor.

5.2 Turning on one modulus

Taking $\xi_3 \neq 0$, the moduli space is now given by

$$M_{11\ldots 1}M_{22\ldots 2}M_{33\ldots 3} - M_{33\ldots 3}\xi_3^n = (-1)^{n-1}M^n_{123}\,. \tag{5.2}$$

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In this moduli space, two of the original fixed lines, given by $M_{11\ldots 1}M_{22\ldots 2} = 0$, (plus $M_{33\ldots 3} = M_{123} = 0$) are deformed to a smooth fixed $\mathbb{C}^*$ at $M_{11\ldots 1}M_{22\ldots 2} = \xi_3^n$, $M_{33\ldots 3} = M_{123} = 0$. This fixed line is the only singularity and again it corresponds to the moduli space for $M = 1$: $\phi_1\phi_2 = \xi_3$ and $\phi_3 = 0$.

To ease the notation, let's rewrite (5.2) as $xyz - z\xi_3^n = t^n$. For a fixed value of $z \neq 0$, this can be written as $xy = \tilde{t}^n + \xi_3^n$, which is a resolution of an A_{n-1} singularity. This resolution can be blown up introducing $n-1$ independent homology 2-cycles and a large set of supersymmetric $\mathbb{P}^1$'s. All this is fibered over z and degenerates at the fixed line, a picture which suggests that on the total space each of these $\mathbb{P}^1$'s corresponds to a 3-cycle.

5.3 ξ_2, $\xi_3 \neq 0$

There are no singularities in the $M = n$ moduli space. This matches with the fact that there are no solutions for the $M = 1$ case. Recall that the analysis is local: what has happened is that the singularities have been sent to infinity.

5.4 $\xi_i \neq 0$

When all the moduli are turned on, there are $n-1$ singularities on the $M = n$ moduli space. On the other hand, there are two solutions for $M = 1$: $\phi_i = \pm(\xi_j\xi_k/\xi_i)^{1/2}$ (with the same sign taken for all i). In this case, the correspondence we have observed so far between singularities of the $M = n$ moduli space and solutions of $M = 1$ is no longer evident.

In the appendix we argue that for the regular representation there is only a finite number of solutions with $M_{\underbrace{ii\ldots i}_{k}} \neq 0$, for $k < n$. We can display $n+1$ of them (actually $M+1$ of them, for $M \leq n$)

$$\phi_i = +\left(\frac{\xi_j\xi_k}{\xi_i}\right)^{1/2} \mathrm{I}_{M-p,p} \qquad \mathrm{I}_{M-p,p}{=}\mathrm{diag}(\underbrace{+,+\ldots+}_{M-p},\underbrace{-,-,\ldots-}_{p}). \tag{5.3}$$

These $n+1$ special solutions (the only commuting ones) seem to correspond to the different possibilities of distributing n objects between the two solutions of $M = 1$. The gauge invariant polynomials for the special solutions are

$$M_{ii\ldots i} = \frac{n-p+(-1)^n p}{n}\left(\frac{\xi_j\xi_k}{\xi_i}\right)^{n/2} \qquad M_{123} = \frac{n-2p}{n}\left(\xi_1\xi_2\xi_3\right)^{1/2} \tag{5.4}$$

and for $M = n$ generically don't coincide with the singularities.

6. Fractional branes

We now discuss the interpretation of the Coulomb branch of the gauge theory. For p-branes with $p < 3$ the world-volume theory is the naive dimensional reduction of the $d = 4$ theory we described and contains scalar partners of the gauge field, which we write as the real adjoint fields X^i. Their potential will be minimized by any vevs with $[X^i, X^j] = 0$ and if $\zeta = 0$ and $\phi = 0$ this is a supersymmetric vacuum with unbroken $\mathrm{U}(1)^n$.

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Not only does this branch have n moduli describing positions in the space transverse to the orbifold but it is clear from world-sheet considerations that it looks like a gravitational and RR source of strength $1/n$ (compared to the original p-brane) at each of these points. Thus the Coulomb branch describes fractional branes bound to the fixed point, just as for the case without discrete torsion [10]. In that case these were interpreted as branes wrapped around hidden two-cycles. However, the singularities with discrete torsion do not contain two-cycles in the usual string theory sense (we did not see the corresponding closed string states) so the story must be rather different here.

The most obvious difference is that, since there is a unique irreducible projective representation, the fractional branes are classified by a single conserved $\mathbb{Z}_n$ quantum number. This shows up in the fact that the gauge theory is labelled by the single integer M, and if $M \geq n$ we can find mixed Coulomb-Higgs branches embedding the Higgs branch in any $\mathrm{U}(n)$ subgroup of $\mathrm{U}(M)$. Thus n fractional branes can annihilate to form a conventional brane.

The appearance of fractional branes in these models raises questions concerning their interpretation, their location and their charge. Let us raise these issues in turn.

6.1 Interpretation

In previous cases, fractional branes were interpreted as wrapped $p+2$ branes. For the unresolved orbifold and when one fixed line is resolved, this picture makes some sense here. Since the fixed lines are fixed under a single group element, locally the geometry is $\mathbb{C}^2/\mathbb{Z}_n \times \mathbb{C}$ and we expect to see branes wrapped around these hidden two-cycles. After the resolution, one can check that the volume (integral of the holomorphic two-form) of the two-cycles increases with distance from the origin (in the third coordinate), which explains why the branes will then be confined to the neighborhood of the origin.

However these two-cycles are not homology two-cycles so it is not obvious why such wrapped branes should be stable. Indeed there is only a conserved $\mathbb{Z}_n$ quantum number. This strongly suggests that the objects have some interpretation as branes wrapped about torsion 2-cycles, which could make contact with the proposal of [11].

This proposal had two parts. The first point was that for geometric compactifications, discrete torsion is naturally identified with $\int B$ over the torsion part of the 2-cohomology, or more precisely with $\mathrm{Hom}(H^2(M), \mathbb{C}^*)$. This is in agreement with our suggested interpretation, as the $\int B \wedge C^{(1)}$ on the D2-brane would lead to the correct fractional D0-charge.

The second part of their proposal was that orbifolds with discrete torsion would admit natural resolutions (possibly not Calabi-Yau) which exhibited this torsion explicitly. We have nothing to say about this beyond the comment that if so, it should also be true for our resolved moduli spaces.

In light of the connection between D-brane charges and K-theory [12], we can contemplate the possibility that the presence of these torsion 2-cycles can be confirmed by studying the K-theory of the moduli space, rather than its cohomology (anyway, recall that we are being a little bit cavalier with the use of the word "cohomology" for our varieties).

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As was pointed out in [13], independently of the details of the variety, in the presence of discrete torsion the available susy cycles are three-cycles, but we don't have $D(p+3)$-branes to wrap about these cycles! Take IIa for definiteness; the natural objects analogous to the fractional branes of [10] in this situation would be strings coming from wrapped D4-branes. The three-cycles are not localized to the fixed points but instead are the two-cycles of the resolved $\mathbb{C}^2/\mathbb{Z}_n$ fixed lines, suspended over a real line connecting two conifold singularities.

It is not surprising that we did not see these objects in the framework discussed here but we are left with the interesting question: can we develop a gauge theory (or other construction) of these strings ? Perhaps a D2-brane with one dimension stretched along a fixed line could be deformed into two of these objects.

6.2 Location of fractional branes

Recall that the moduli space for $M = 1$ with all ξ non-zero consisted of two points $\phi_i^{(\pm)} = \pm(\xi_j\xi_k/\xi_i)^{1/2}$. Thus there are two physically (though not topologically) distinct elementary fractional branes.

One can compute the values of the invariant polynomials for these configurations and intuitively, one might have expected these to be identifiable with singularities of the moduli space for the regular representation. Since there are $n-1$ singularities and two fractional branes, this intuition already has problems. In fact the invariants corresponding to these points are not on the generic branch at all!

This situation persists for combinations of fractional branes. Let us consider n fractional branes, for which there are $n+1$ distinct combinations of the two solutions $\phi^{(\pm)}$. Only for n even and equal numbers N_+ and N_- of $\phi^{(+)}$ and $\phi^{(-)}$ is this point on the generic branch (and then it is a singularity).

Thus the decay of the n fractional branes to a regular brane almost always encounters a potential barrier. Another way to see this is to compute the value of the superpotential $W(\phi)$ at the various solutions. If we consider domain wall solutions of the four dimensional gauge theory interpolating between the two solutions (or reduce to two dimensions), they will be BPS with central charge equal to the difference $W(\phi_a) - W(\phi_b)$. It is easy to check that the Higgs branch has $W = 0$, while the fractional brane solutions have $W = \sqrt{\zeta_1\zeta_2\zeta_3}(N_+ - N_-)$.

Another interesting definition of the 'location' of the fractional branes would be to consider a larger theory containing both a regular and a fractional brane, and find the values of the regular brane moduli for which additional massless states appear in the system. We leave this as a problem for future research.

The picture looks very non-geometric, on any scale comparable with ζ, not just at the singularity. This leads to potential paradoxes in the large R_{11} limit as we can take $\sqrt{\zeta} \gg l_{p11} = g_s^{1/3} l_s$, and at these scales we expect that M theory is geometric. It is not too clear if these are actual paradoxes. For one thing, we did not compute metric data and it might be that the $O(\sqrt{\zeta})$ separation in complex coordinates does not translate into an $O(\sqrt{\zeta})$ distance. Even if it does, if we try to take the large R_{11} limit a la Matrix theory, the large N limit might produce a very different picture. It would be interesting to find some geometric picture which can evolve smoothly to the one presented here.

6.3 Charge quantization

Fractional branes carry $1/n$ of the charge of the original branes. At first sight this seems to conflict wih the Dirac quantization condition for D-brane charges. In models without discrete torsion, a way which has been proposed out of this paradox is by analogy with the answer to the familiar question of why the fractional charge of quarks is not inconsistent with the Dirac quantization condition for monopoles [14]. In both cases the fractionally charged objects carry an additional charge, and that can modify the Dirac quantization condition. In the case of quarks this charge is of course color, and if the group is $\mathrm{U}(1)_{EM} \times \mathrm{SU}(N)_c$ the quantization condition is $eg = 2\pi n/N$. For fractional branes, both in the ALE case and in the $T^6/\mathbb{Z}_n \times \mathbb{Z}_n$ without discrete torsion, from $h^{1,1} \neq 0$ we see that there are 2-cycles, and the original 3-form is reduced on them to a 1-form. Fractional branes are charged under this reduced 1-form [10], and that could solve the paradox.

However, in the present case, this cannot be the resolution, as there are no RR gauge fields in twisted sectors for the objects to couple to. Consider D0's in IIa for definiteness; we found that all the twist states corresponded to elements of $H^{1,2}$, whose RR partners are scalars.

Checking the Dirac condition requires introducing the D6-brane and either computing properly normalized charges for the two objects or else computing their interactions directly (perhaps from the annulus diagram, along the lines of [15]). The first computation involves the volume of the internal space and when passing to the orbifold, gives a factor of the order of the group which could resolve the problem. However, an indication that something more unusual is going on here is that the order of the group is n^2, compared with the $1/n$ we are trying to account for.

The second computation can be done easily if we take the open string (gauge theory) point of view and restrict to the massless sector. In [16] it was shown (in a very similar matrix theory problem) that the magnetic monopole interaction between a Dp and D$(6-p)$ brane can be derived from the Berry phase of the hamiltonian describing fermionic strings stretched between these objects. These fermions are a doublet of the SO(3) transverse to both branes and their hamiltonian is simply $H = \vec{\sigma} \cdot \vec{X}$; it is well known that the Berry phase for this hamiltonian is described by the magnetic monopole gauge potential. On the other hand, massive string modes will always come in pairs with cancelling Berry phase. In particular we can ignore the winding states around T^6 for this computation.

Thus the interaction relevant for the Dirac quantization condition can be computed simply by counting fermionic strings. It also means that perturbative consistency of string theory guarantees that D-branes will satisfy the Dirac rule.

The matrix $N_{ab} = \mathrm{Tr}_{ab}\ (-1)^F$ counting massless Ramond doublets (with chirality) between the a'th and b'th brane is the CFT analog of the intersection form in a geometric compactification. It is an index and thus will not vary under continuous deformations. In the geometric case, if we consider a set of branes wrapping an integral homology basis, this form must be integral and unimodular by Poincaré duality, proving that the Dirac condition is saturated. By computing N_{ab} in a CFT compactification we can check that a particular set of D-branes also saturates the Dirac condition and is thus complete.

D6-branes wrapped about the internal space in these orbifold theories are defined by the same projection (3.1) of flat space gauge theory but now with the space-time operation R acting on the world-volume coordinates as well. If we start with an N-dimensional representation γ this will lead to a $\mathrm{U}(N)$ gauge theory on the quotient space T^6/Γ with specific boundary conditions. depending on the choice of representation.

The 0–6 strings in our problem will then give fermionic fields χ satisfying the projection

$$\left(\gamma^{(6)}(g)\right)^{-1} \chi\gamma^{(0)}(g) = R(g)\chi\,. \tag{6.1}$$

In fact $R(g) = 1$ here (in contrast to the vertex operators $V(\zeta)$ we discussed earlier which do transform under $\mathrm{U}(1)^3$ rotations). This is clear because the Ramond states must form a representation of the SO(6) rotating the DN directions before applying the orbifolding; as is well known this is a singlet representation.

Let us briefly discuss the case without discrete torsion first (we will return to this elsewhere). We first note that in contrast to the 0-branes it is more reasonable to call the object with $N = 1$ the elementary 6-brane since its world-volume theory is U(1) gauge theory on the quotient space. Both it and the fractional 0-brane will carry one-dimensional representations of Γ and the projection (6.1) implies that only if the two branes carry the same representation γ will a 0-6 string survive: we have $N_{ai} = \delta_{a,i}$ in this basis. Thus the Dirac condition is saturated if we include the $N = 1$ 6-branes.

For the case with discrete torsion, the minimal theory then takes $\gamma^{(0)} = \gamma^{(6)} = \gamma_1$, an $n \times n$ representation, to describe a fractional 0-brane and an elementary 6-brane. A single component $\chi \propto \mathbf{1}$ survives the projection and we conclude that these two objects satisfy the Dirac condition with the minimal flux quantum.

However, the striking feature of this elementary 6-brane is that it is a $\mathrm{U}(n)$ gauge theory on the quotient manifold, not a U(1) theory, since γ_1 is an n-dimensional representation. As we commented above, the equations (3.1) have a solution for generic $\mathrm{U}(n)$ gauge fields in one fundamental region T^6/Γ; they simply determine the corresponding fields in the other fundamental regions.

Thus the final resolution of our paradox turns out to be that, in the case with discrete torsion, the elementary 6-brane is actually an "n-fold bound state," in the sense that it carries $\mathrm{U}(n)$ gauge fields.

7. Conclusions and further questions

We derived D-brane gauge theories for $\mathbb{C}^3/\mathbb{Z}_n \times \mathbb{Z}_n$ orbifolds with discrete torsion and studied the moduli space of a D-brane at a point (say a D0 in IIa). We were able to find a simple exact equation for this moduli space as a subvariety of $\mathbb{C}^4$. In agreement with expectations the closed string moduli deform the moduli space and resolve the fixed lines but do not allow fully resolving the singularity.

However the detailed results for $n > 2$ conflict with the intuition that discrete torsion, being non-geometrical, had to be "located" at a particular singularity. Instead we find $n-1$ conifold singularities separated (in complex parameter space) by $O(\sqrt{\zeta})$, where ζ is the scale of the resolution parameter.

Was there any real basis for the intuition that discrete torsion would be concentrated at one singularity? Probably not. One argument against this is that, in the global context (consider $T^6/\mathbb{Z}_3 \times \mathbb{Z}_3$ for example), the value of discrete torsion is not independently adjustable at each of the singularities but rather is a global choice to be made for the entire orbifold. From this point of view any number of singularities might have appeared in the resolution with discrete torsion being a global invariant not necessarily detectable at any one of them.

We also found fractional branes which are BPS and carry a conserved $\mathbb{Z}_n$ quantum number. Some of their properties are consistent with the idea that these are $p+2$-branes wrapped about a zero volume torsion 2-cycle of the type suggested in [11]. Other properties — there are two elementary fractional branes distinguished by a non-topological quantum number, and the annihilation of n branes to a regular brane encounters a potential barrier — are rather peculiar.

The $T^6/\mathbb{Z}_2 \times \mathbb{Z}_2$ model is mirror to the same orbifold without discrete torsion. Since this can be completely resolved, this should provide a geometrical picture of the fractional branes in terms of 3-branes. Presumably the potential barrier would mean that a supersymmetric T^3 cycle (mirror to the D0) is homologous to a sum of n supersymmetric cycles without moduli (say S^3's) but that they are connected only by deforming through non-supersymmetric cycles.

The fractional branes carry charge $1/n$ yet satisfy the Dirac quantization condition; related to this, the elementary conjugate $6-p$-brane is a "bound state" in the sense that its world-volume theory is a $\mathrm{U}(n)$ gauge theory.

It will be quite interesting to make contact between these results and the K-theory description of D-brane charge, and to develop either a geometrical interpretation of discrete torsion or perhaps a noncommutative geometric description.

Acknowledgments

B.F. would like to thank Duiliu-Emanuel Diaconescu, Jaume Gomis, and Christian Roemelsberger for useful discussions and encouragement. M.R.D. would like to thank Edward Witten for a discussion.

A. The polynomial equation of the moduli space

In this appendix we prove that on the moduli space of our $\mathrm{U}(n)$ gauge theory, when all $\zeta \neq 0$, any gauge invariant polynomial I can be written as a polynomial in terms of $M_{ii\ldots i}$ and M_{123}: $I = P_I(M_{11\ldots 1}, M_{22\ldots 2}, M_{33\ldots 3}, M_{123}; \zeta_i)$. This means that the moduli space is a subvariety of $\mathbb{C}^4$ defined by polynomial equations, which could be obtained in principle by finding a complete set of relations between gauge invariants and substituting these linear relations.

The simplest relation comes from taking the trace of the F-flatness conditions (3.10): $M_{ij} = \xi_k$.

The idea is, given some invariant $M_{i_1 i_2 \ldots}$, to use the deformed F-flatness conditions (3.10) and the cyclic property of the trace, to shift around one of the indices, until the original ordering is recovered. If this comes with a phase, we can express the invariant in terms of lower degree invariants. The simplest example will illustrate the general method.

$$\begin{aligned}\phi_1\left(\phi_1\phi_2 - \epsilon^{-1}\phi_2\phi_1\right) &= -\zeta_3\phi_1 \Rightarrow M_{112} - \epsilon^{-1}M_{121} = -\zeta_3 M_1 \Rightarrow \\ \left(1-\epsilon^{-1}\right) M_{112} &= -\zeta_3 M_1 \Rightarrow M_{112} = M_1 M_{12}\,.\end{aligned}$$

Another example (which we leave as an exercise for the idle reader) is that $M_{iijj} = M_{ijij} = M_{ij}^2$.

However, this method does not work for every invariant. For invariants with all indices equal, as $M_{11\ldots1}$, the method does not even apply. This is not the only case for which the method fails to express an invariant in term of lower degree invariants: if $M_{ij\ldots k}$ has charge (n_1, n_2, n_3), then when $n_1 - n_2 \equiv n_2 - n_3 \equiv n_3 - n_1 \equiv 0\ (n)$ — i.e. $n_1 \equiv n_2 \equiv n_3\ (n)$ – we obtain instead an identity involving only lower degree invariants. For example, for $n > 2$

$$\begin{aligned}M_{112233} &= e^{-\frac{2\pi i}{n}2} M_{122133} - \left(1+\epsilon^{-1}\right)\zeta_3 M_{1233} \\ &= M_{112233} + \left(1+\epsilon^{-1}\right)\left(\zeta_2 M_{1223} - \zeta_3 M_{1233}\right) \\ &\Rightarrow \xi_2 M_{1223} = \xi_3 M_{1233}\end{aligned}$$

so using this method we fail to express M_{112233} in terms of lower degree polynomials, but instead obtain a relation between lower degree polynomials.

At this point our list of potentially independent polynomials on the moduli space consists of all the polynomials of the form $M_{ii.i}$, and the polynomials $M_{11\ldots122\ldots233\ldots3}$ with $n_1 \equiv n_2 \equiv n_3(n)$. To obtain more relations we assume that our matrices are n dimensional and use the Cayley-Hamilton theorem. This states that a matrix ϕ will solve the equation

$$P_\phi(\phi) = 0\,, \tag{A.1}$$

where $P_\phi(x)$ is the characteristic polynomial of ϕ, a polynomial of degree n whose coefficients are polynomial in $\operatorname{Tr}\phi^k$ for $k \leq n$. The simplest proof follows from the fact that diagonalizable matrices are a dense subset.

Since (A.1) is a matrix relation we have

$$\operatorname{tr} A P_\phi(\phi) = 0 \tag{A.2}$$

for any matrix polynomial A. This shows that any invariant of degree greater than n with n repeated indices can be written in terms of lower degree invariants. This reduces our list of potentially independent invariants to $M_{ii\ldots i}$ with up to n indices plus $M_{123123\ldots123}$ with strictly less than $3n$ indices.

We can do even better. Now we are going to argue that the situation is the following: for generic values of $M_{ii\ldots i}$, when the relation $\xi_1^n M_{\underbrace{11\ldots1}_{n}} = \xi_2^n M_{\underbrace{22\ldots2}_{n}} = \xi_3^n M_{\underbrace{33\ldots3}_{n}}$ is not satisfied, we can prove that $M_{\underbrace{ii\ldots i}_{k}} = 0$ for $k < n$. On the other hand, in the special case when that relation is satisfied, $M_{ii\ldots i}$ for $k<n$ need not to be zero, (and indeed it is not zero for some solutions), but we nonetheless argue that $M_{ii\ldots i}$ can only take a finite number of values, so they cannot parametrize a continuous direction.

To start with, one can very easily prove that

$$M_{\underbrace{11\ldots1}_{a}\underbrace{22\ldots2}_{b}\underbrace{33\ldots3}_{c}}(e^{\frac{2\pi i}{n}b}-e^{\frac{2\pi i}{n}c})=$$
$$=\left[\xi_3(e^{\frac{2\pi i}{n}b}-1)M_{\underbrace{11\ldots1}_{a-1}\underbrace{22\ldots2}_{b-1}\underbrace{33\ldots3}_{c}}-\xi_2(e^{\frac{2\pi i}{n}c}-1)M_{\underbrace{11\ldots1}_{a-1}\underbrace{22\ldots2}_{b}\underbrace{33\ldots3}_{c-1}}\right].$$

Applying this first to $M_{\underbrace{11\ldots1}_{k}\underbrace{22\ldots2}_{k}\underbrace{33\ldots3}_{k}}$ and then succesively to the resulting relations we obtain that for $k<n$

$$\xi_1^k M_{\underbrace{11\ldots1}_{k}}=\xi_2^k M_{\underbrace{22\ldots2}_{k}}=\xi_3^k M_{\underbrace{33\ldots3}_{k}}. \tag{A.3}$$

On the other hand, take ϕ in (A.2) to be ϕ_1 and A in to be successively $\phi_2\xi_1^{n-1}$, $\phi_2^2\xi_1^{n-2}$,... $\phi_2^{n-1}\xi_1$, and in each case then substract the same relation with $1\leftrightarrow 2$. Using (A.3) , we obtain

$$M_{\underbrace{11\ldots1}_{n}}M_{\underbrace{22\ldots2}_{k}}\xi_1^{n-k}=M_{\underbrace{22\ldots2}_{n}}M_{\underbrace{11\ldots1}_{k}}\xi_2^{n-k}. \tag{A.4}$$

Now if any of the $M_{11\ldots1}$ with less than n indices is not zero, we can use the last relation (and two similar ones, replacing $1\to 3$ in the first, and $2\to 3$ in the second) to prove that $\xi_1^n M_{\underbrace{11\ldots1}_{n}}=\xi_2^n M_{\underbrace{22\ldots2}_{n}}=\xi_3^n M_{\underbrace{33\ldots3}_{n}}$. This shows that for generic values of $M_{\underbrace{ii\ldots i}_{n}}$ and ξ_i, (that is, when the previous relation is not satisfied), we have $M_{\underbrace{ii\ldots i}_{k}}=0$ for $k<n$. What can we say when $\xi_1^n M_{\underbrace{11\ldots1}_{n}}=\xi_2^n M_{\underbrace{22\ldots2}_{n}}=\xi_3^n M_{\underbrace{33\ldots3}_{n}}$? Plugging this relation and (A.3) into the characteristic polynomials of ϕ_1, ϕ_2 and ϕ_3 we learn that in this case $\phi_2=(\zeta_1/\zeta_2)g\phi_1 g^{-1}$ and $\phi_3=(\zeta_1/\zeta_3)\tilde{g}\phi_1\tilde{g}^{-1}$. Anyway, we can argue that there is only a finite number of values the $M_{\underbrace{ii\ldots i}_{k}}$ can take. To see this take ϕ in (A.2) to be ϕ_1 and A to be succesively ϕ_2, $\phi_2^2\ldots\phi_2^{n-1}$. Using (A.3) we obtain $n-1$ equations for $M_{ii\ldots i}$, and this system of equations has a finite number of solutions.

Finally, we have to deal with the polynomials of the kind $M_{123\ldots123}$. We can not apply (A.2) as it stands, since we have argued that for generic values of ξ_i, $M_{ii\ldots i}=0$ for $k<n$, so the Cayley-Hamilton theorem reduces to $\phi_i^n-M_{11\ldots1}I=0$. If we consider the characteristic polynomial of $\phi_1\phi_2$ it does not work neither. If we consider the characteristic polynomial of $\phi_1\phi_2\phi_3$ and take $A=\phi_1,\phi_1^2\ldots$, we obtain relations for these polynomials. FOr instance, for $n=3$ we obtain

$$M_{123123}=M_{123}^2+2\epsilon(1-\epsilon)M_{12}M_{31}M_{23}. \tag{A.5}$$

So for generic values of M_{ij} we see that $M_{123\ldots123}$ is also determined in terms of lower degree polynomials. This concludes our argument that on the moduli space, all the non-zero gauge invariant polynomials can be written in terms of 4 polynomials: $M_{\underbrace{11\ldots1}_{n}}$, $M_{\underbrace{22\ldots2}_{n}}$, $M_{\underbrace{33\ldots3}_{n}}$ and M_{123}.

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